Algebraic Models for Covering Arrays

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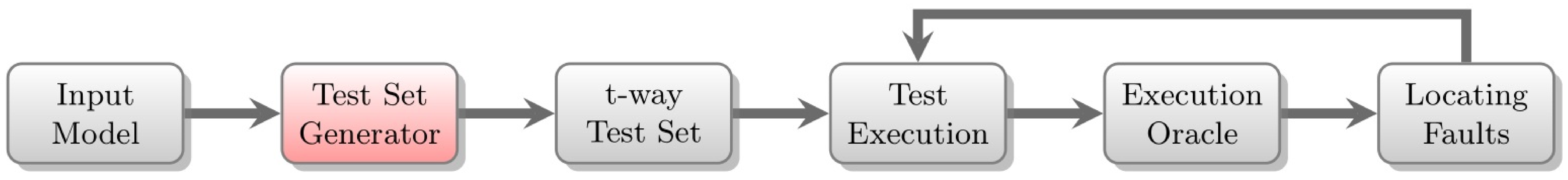
Covering Arrays and Optimal Covering Array Generation

Covering Arrays for Combinatorial Testing

- Covering Arrays (CAs) provide the theoretical means for Combinatorial Testing (CT).
- Columns of a CAs map to the parameters of a system under test.
- Rows of a CA encode the individual test cases.
- CAs guarantee that derived test sets **cover** all ***t*-way interactions**.
- Smaller test suites and reduced testing costs can be achieved by constructing CAs with less rows.



The Optimal Covering Array Generation Problem

- Given a **strength t**, a number of columns k and the respective column alphabet size v.
- Construct a covering array CA($N; t, k, v$) with the smallest number of rows N.
- Direct constructions for optimal CAs, i.e., minimal N, exist only for boundary and some special cases.
- The general problem of actually constructing **optimal CAs** remains unsolved.

Algebraic Modelling of Covering Arrays

Definition of Covering Arrays

- Defining property of a CA($N; t, k, v$):
  - For any sub-matrix comprised by t different columns it holds that
  - All $\{0, \dots, v-1\}^t$ t-tuples appear at least once as a row

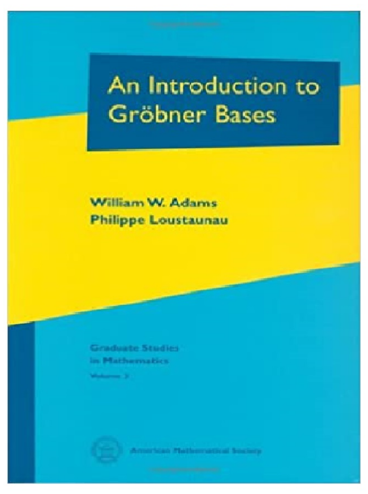
Covering Arrays as Solutions of Equation Systems

- By virtue of an appropriate algebraic structure, e.g.:
  - Integral domain R with unity
  - That has t linearly independent elements, when interpreted as $\mathbb{Z}_v$-module

Theorem. Let R be a ring and $(R, a_1, \dots, a_t)$ have the v-ary t-way interaction distinguish property, and $X := (x_{i,j})$ be an $N \times k$ array of variables. Then any solution to the following system of equations in the unknowns $x_{i,j}$ yields a CA($N; t, k, v$):

1. $\forall i \in \{1, \dots, N\}, \forall j \in \{1, \dots, k\}$
$$\prod_{r=0}^{v-1} (x_{i,j} - r) = 0. \tag{1}$$
2. $\forall C \in \binom{\{k\}}{t},\ \forall (u_1, \dots, u_t) \in [v]^t$:
$$\mathbf{prod}(X \cdot \iota^C_{t,k}(a_1, \dots, a_t) - \mathbf{1} \cdot (u_1, \dots, u_t) \cdot (a_1, \dots, a_t)^T) = 0. \tag{2}$$

Algebraic and Symbolic Methods



$$\begin{pmatrix} 0 & 0 & x_1 \\ 1 & 0 & x_2 \\ 0 & 1 & x_3 \\ 1 & 1 & x_4 \end{pmatrix} \cdot \begin{pmatrix} a \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} bx_1 \\ bx_2 + a \\ bx_3 \\ bx_4 + a \end{pmatrix}$$

```
S:=RationalField();
P:=PolynomialRing(S,k*N+2);
R:=PolynomialRing(S,k*N);
M := ZeroMatrix(P,k,N);
for i in [1..k] do
    for j in [1..N] do
        M[i][j] := P.((i-1)*N+j);
    end for;
end for;
```

$$\begin{aligned}
x_1(x_1-1) &= 0\\
x_2(x_2-1) &= 0\\
x_3(x_3-1) &= 0\\
x_4(x_4-1) &= 0\\
(bx_1-b)(bx_2+a-b)(bx_3-b)(bx_4+a-b) &= 0\\
(bx_1-b)(bx_2-b)(bx_3+a-b)(bx_4+a-b) &= 0\\
(bx_1-a-b)(bx_2-b)(bx_3-a-b)(bx_4-b) &= 0\\
(bx_1-a-b)(bx_2-a-b)(bx_3-b)(bx_4-b) &= 0\\
b^2x_1(bx_2+a)x_3(bx_4+a) &= 0\\
b^2x_1x_2(bx_3+a)(bx_4+a) &= 0\\
(bx_1-a)b^2x_2(bx_3-a)x_4 &= 0\\
(bx_1-a)(bx_2-a)b^2x_3x_4 &= 0
\end{aligned}$$

- Multiple approaches to solve the equation systems:
  - Algebraic Solvers (Gröbner Bases)
  - SAT-Solvers
  - Search techniques and high performance computing

Algorithmic Formulation for Covering Arrays

- The algebraic model allows to formulate an algorithm for CA computation.
- Above theorem yields the following algorithm.

Algorithm 1 ALGEBRAICSEARCHCAS

INPUT: N, t, k, v
Require: $t \leq k$
Create a symbolic $N \times k$ array X containing variables $x_1, \dots, x_{Nk}$
$EQall := \emptyset$
for $C \in \binom{\{k\}}{t}$ **do** ▷ Add coverage equations
  for $\mathbf{u} \in [v]^t$ **do**
    $EQ := \mathbf{prod}(X \cdot \iota^C_{t,k}(a_1, \dots, a_t) - \mathbf{1} \cdot (u_1, \dots, u_t) \cdot (v^0, \dots, v^{t-1})^T) = 0$
    add EQ to $EQall$
  end for
end for
for $i = 1, \dots, Nk$ **do** ▷ Add domain equations
  $EQ := \prod_{j=0}^{v-1}(x_i - j) = 0$
  add EQ to $EQall$
end for
Interpret $EQall$ as subset of $\mathbb{Q}[x_1, ..., x_{Nk}]$
V = SOLVE($EQall$) ▷ Call external solver
if $V \neq \emptyset$ **then**
  return V;
else print "No CA exists";
end if

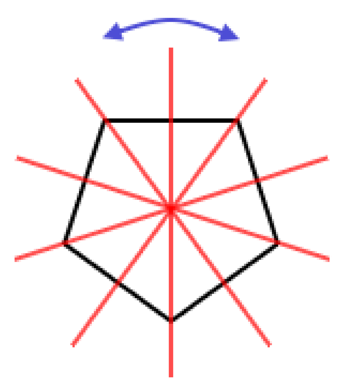
Computational Results

- The approach is exact and complete in nature.
- Full enumeration of **all optimal CAs** for specific parameters.

CA instance	Solver	# Vars	# Sols	CAN
$CA(4; 2, 3, 2)$	GB	12	48	4
$CA(5; 2, 3, 2)$	GB	15	1440	4
$CA(5; 2, 4, 2)$	GB	20	1920	5
$CA(8; 3, 4, 2)$	GB	32	80640	8
$CA(9; 2, 3, 3)$	C/MPI	27	$\geq 3 \cdot 10^6$	5

Future Work

- Enhancement of existing work by structuring equations.
- Symmetry breaking during solving process.
- Algebraic modellings of related designs.
- Hybridization with other approaches.



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